Truthlikeness of stratified theories

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TRUTHLIKENESS OF STRATIFIED THEORIES

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1. *Introduction*

At the end of my "A structuralist approach to truthlikeness" in this volume I stressed that even the approximative notions of comparative and quantitative theoretical truthlikeness were introduced in a *naive* structuralist context, viz. without a distinction between a, relatively, observational and a theoretical level. Moreover, I pointed out that in such a context there would be no serious role for theories other than those suggested by the internal and external evidence, let alone for non-evidence-saving theories.

In this paper I shall extend the truthlikeness analysis to the situation that there are at least two levels. I'll concentrate on (approximative) *comparative* truthlikeness, because, as in the case of methodological consequences, quantitative truthlikeness does not seem to lead to forceful bridgetheorems.

After restating the main aspects of one-level-truthlikeness in Section 2, the main truthlikeness relations between two levels will be given in Section 3. Finally Section 4 is an attempt to characterize the generality of the analysis. The rest of the present section gives the preliminaries for dealing with truthlikeness in the context of two levels.

Let M_t indicate the set of all structures of a certain (similarity-)type, representing the *conceptual-theoretical* possibilities in the context of interest, with x, y, z, ... as variables for elements and A, B, ... as variables for subsets.

Let M_o indicate the set of all structures of a certain type, representing the *conceptual-observational* possibilities of the context, $|M_o|>1$, with u, v, w, ... as variables for elements and F, G, ... as variables for subsets.

In addition I assume that there is a reduction-, restriction- or *projection-function* p from M_t *onto* M_o, assigning to every $x \epsilon M_t$ the unique $u \epsilon M_o$ (hence, $p(x) = u$), such that u is a substructure of x. The projection of A is indicated by pA (i.e. $pA =_{df} \{u \epsilon M_o | \exists x \epsilon A\ p(x) = u\}$). As done before, I indicate the complement of A, relative to M_t, hence M_t-A, by a similar device: kA; similarly, M_o-F = kF. It will be important to have the following relational property in mind in the sequel.

(1.1) $pkA \supseteq kpA$, hence, also $kpkA \subseteq pA$

Note that $u \in kpkA$ means that there is no $x \in A$ such that $p(x) = u$, i.e. u has only "originals" in A, none outside A. Of course, many A's will be such that $kpkA = \emptyset$, but not all. The extreme cases are as follows:

(1.2) $kpkA = \emptyset$ iff $\forall u \in M_o \exists x \in A\ p(x) = u$
(1.3) $kpkA = pA$ iff $\forall u \in pA\ \forall x \in M_t\ (p(x) = u \Rightarrow x \in A)$

From now on, kpkA will be abbreviated by p^*A.

Next I assume structurelikeness notions c_t and c_o, underlying M_t and M_o, respectively. Both ternary relations are assumed to be reflexive, transitive and strongly self-similar, i.e. for instance for c_t:

$c_t(x;y,y)$,
$c_t(x;y,z)$ & $c_t(x;z,z') \Rightarrow c_t(x;y,z')$, and
${}^+c_t(x;x,y)$, respectively.

Here, $c_t(x;y,z)$ is to be read as "y is as close to x as z" and ${}^+c_t(x;y,z)$ is defined as "$c_t(x;y,z)$ & not-c_t $(x;z,y)$". Note that strong self-similarity is equivalent to the conjunction of $c_t(x;x,y)$ ((weak) self-similarity) and $c_t(x;y,x) \Rightarrow y{=}x$ (centering).

It seems plausible to assume the following relation between c_t and c_o:

c-conservation: $c_t(x;y,z) \Rightarrow c_o(p(x);p(y),p(z))$

Extreme cases of c_t and c_o are the so-called trivial or naive structurelikeness notions: c_t is naive if $y \neq x \neq z$ implies $c_t(x;y,z)$, the naive c_t is indicated by $c_t{}^n$. The naive c_o is defined similarly, and indicated by $c_o{}^n$.

The following theorem shows the (strong) consequences of applying the conservation-assumption to $c_t{}^n$:

(2) c-conservation and $c_t = c_t{}^n$ imply $c_o = c_o{}^n$ and p is a one-one-function.

The proof of $c_o = c_o{}^n$ is straightforward. The proof that p is injective is as follows. Suppose $x \neq y$ and $p(x) = p(y) = u$. Let $z \neq x$, then $c_t{}^n(x;z,y)$; hence, by c-conservation, $c_o{}^n(u;p(z),u)$; hence, from strong self-similarity, $p(z) = u$; hence $pM_t = \{u\}$, but p is *onto* M_o, i.e. $pM_t = M_o$ and $|M_o|>1$, hence impossible; q.e.d.

Note that an interesting two-level situation requires that p is not a one-one-function. Hence, naive truthlikeness, which was based on trivial structurelikeness, does not have room for interesting relations of truthlikeness between different levels.

2. *Truthlikeness, restricted to one level*

Let $X_t = X \subseteq M_t$ indicate the set of empirical possibilities relative to M_t, called the *theoretical possibilities*. We repeat the main definitions and theorems, starting with the *approximative* notions:

(i) $Ca^{in}(X;A,B) \quad =_{df} \forall x \epsilon X \forall y \epsilon B \exists z \epsilon A \;\; c_t(x;z,y)$
(ii) $Ca^{ex}(X;A,B) \quad =_{df} Ca^{in}(kX;kA,kB)$
(iii) $Ca(X;A,B) \quad =_{df} Ca^{in}(X;A,B) \;\&\; Ca^{ex}(X;A,B)$

The *naive* notion C^{in} is defined by

(i)^n $C^{in}(X;A,B) \quad =_{df} X\text{-}A \subseteq X\text{-}B$

and C^{ex} and C are based on C^{in} in the same way as Ca^{ex} and Ca are based on Ca^{in} by (ii) and (iii), now indicated by (ii)^n and (iii)^n.

The following theorems relate the naive and sophisticated notions

(3) $Ca^{(in/ex)}(X;A,B) \quad \Rightarrow C^{(in/ex)}(X;A,B)$ (Ca implies C)
(4) $c_t = c_t^{\,n} \;\&\; \emptyset \neq A \neq M_t \& \;\; C^{(in/ex)}(X;A,B) \Rightarrow Ca^{(in/ex)}(X;A,B)$
(Ca reduces to C for trivial c_t)

Of course it is possible to repeat the whole story for the observational level M_o, but I shall not write this out. If I want to refer to an M_o-definition or -theorem, e.g. the one corresponding to (ii)/(4), I shall write $(ii)_o$, $(4)_o$, respectively. Such a device will nor be necessary for truthlikeness statements, for their level will always be clear.

On the observational level "the role of X" is played by the subset $X_o \subseteq M_o$ of empirical possibilities relative to M_o, called the *observational possibilities*.

Moreover, in a two-level situation, empirical success is of course *observational success*. Hence, let R indicate the realized observational possibilities, and S the strongest observational law, based on R. Assuming the idealization $R \subseteq X_o \subseteq S$, it is possible to prove the following *backward success-theorems*.

(5.1) $Ca^{in}(X_o;U,V) \Rightarrow Ca^{in}(R;U,V)$
(5.2) $Ca^{ex}(X_o;U,V) \Rightarrow Ca^{ex}(S;U,V)$

Of course, (5) is provable for any $X_o = W$ such that $R \subseteq W \subseteq S$, and this will implicitly be used later.

3. *The main relations*

The primary objective of the present paper is the relation between truthlikeness on both levels. We start by noting that the levels are only properly related if $pX_t = pX = X_o$, called *X-conservation*, which will, besides c-conservation, be assumed throughout this section.

By consequence, the *theoretical claim* of a theory A, viz. "A=X" implies the *observational claim* "pA=X_o".

The following theorems show that (comparative) truthlikeness on the theoretical level is nicely related to truthlikeness on the observational level (p*A = kpkA!).

Th. 1.1 $Ca^{in}(X;A,B) \Rightarrow C^{in}(pX;pA,pB)$
1.2 $Ca^{ex}(X;A,B) \Rightarrow Ca^{ex}(p^*X;p^*A,p^*B)$

Th. 1.1 follows straightforwardly from the definitions (i), $(i)_o$ and c-conservation. For a direct proof of the extreme, naive case of Th. 1.1., it is necessary to use consequence (2) of c-conservation. Th. 1.2 follows by applying (ii), Th. 1.1. and $(ii)_o$, respectively.

From (3) and Th. 1 we obtain directly

(6.1) $Ca^{in}(X;A,B) \Rightarrow C^{in}(pX;pA,pB)$
(6.2) $Ca^{ex}(X;A,B) \Rightarrow C^{ex}(p^*X;p^*A,p^*B)$

Although proof-attempts make it directly clear that we may not substitute "p*" for "p" in Th. 1.1, nor "p" for "p*" in Th. 1.2, these substitutions are allowed in (6.1) and (6.2), respectively. That is, it is possible to prove:

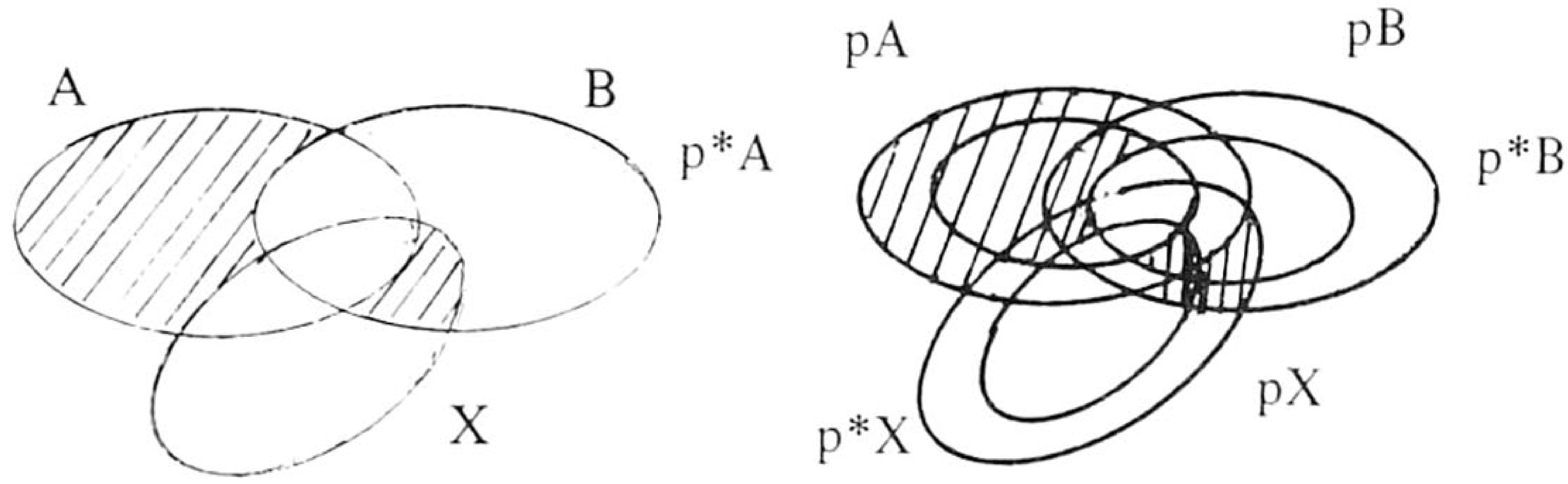
(7.1) $Ca^{in}(X;A,B) \Rightarrow C^{in}(p^*X;p^*A,p^*B)$
(7.2) $Ca^{ex}(X;A,B) \Rightarrow C^{ex}(pX;pA,pB)$

In order to prove (7) it is, due to (3), enough to prove

Th. 2.1 $C^{in}(X;A,B) \Rightarrow C^{in}(p^*X;p^*A,p^*B)$
2.2 $C^{ex}(X;A,B) \Rightarrow C^{ex}(pX;pA,pB)$

Note first that Th. 2.1 can be obtained from Th. 2.2 by applying (ii) and $(ii)_o$ (and kkF=F). Hence, it remains to prove Th. 2.2, which is, according to $(i)^n$ and $(ii)^n$ and after some set-manipulations, equivalent to $A\text{-}X \subseteq B \Rightarrow pA\text{-}pX \subseteq pB$, and this is a straightforward consequence of projection.

From (3), (6) and (7) it follows that Ca(X;A,B) has ((1+2)+(1+2)=)6 "naive", i.e. purely settheoretic, consequences: the shaded areas in the following diagram are empty.



Finally, it is of course possible to attach the success-theorem (5) at the relevant places. For this, it is important to note that, though the necessary idealization $R \subseteq pX \subseteq S$ ($pX = X_o!$) implies, by (1.1), that p^*X is a subset of S, it does not imply that R is a subset of p^*X.

The following scheme gives the total result of downward and backward relations between the internal notions:

$$
\begin{array}{ccccccc}
\text{approximative} & & & \text{naive} & & & \text{approximative} \\
\mathrm{Ca}^{in}(X;A,B) & \overset{(3)}{\Rightarrow} & & \mathrm{C}^{in}(X;A,B) & & \overset{(3)}{\Leftarrow} & \mathrm{Ca}^{in}(X;A,B) \\
\text{Th.1} \Downarrow & & \text{Th.1} \;\swarrow & & \searrow \; \text{Th.2} & & \not\Downarrow \\
\mathrm{Ca}^{in}(pX;pA,pB) & \overset{(3)}{\Rightarrow} & \mathrm{C}^{in}(pX;pA,pB) & \Leftarrow\!/\!\Rightarrow & \mathrm{C}^{in}(p^*X;p^*A,p^*B) & \overset{(3)}{\Leftarrow} & \mathrm{Ca}^{in}(p^*X;p^*A,p^*B) \\
(5) \Downarrow & & (5) \Downarrow & & \Downarrow (5) & & \Downarrow (5) \\
\mathrm{Ca}^{in}(R;pA,pB) & \overset{(3)}{\Rightarrow} & \mathrm{C}^{in}(R;pA,pB) & \Leftarrow\!/\!\Rightarrow & \mathrm{C}^{in}(R \cap p^*X;p^*A,p^*B) & \overset{(3)}{\Leftarrow} & \mathrm{Ca}^{in}(R \cap p^*X;p^*A,p^*B)
\end{array}
$$

The corresponding scheme for the external notions can be obtained by (substituting everywhere "ex" for "in", and) interchanging the upperleft vertical arrow and the upperright vertical "non-arrow", and by replacing "R" as well as "$R \cap p^*X$", in the bottomline, simply by "S".

The main thing that can be said about the opposite directions, forward and upward, is of course that one may not come in conflict with the downward-backward relations. If one concentrates on strict notions (e.g. $^+\mathrm{Ca}(X;A,B) =_{df} \mathrm{Ca}(X;A,B)$ & not-$\mathrm{Ca}(X;B,A)$) one obtains

(8.1) $^+\mathrm{Ca}^{in}(R;pA,pB) \qquad \Rightarrow$ not-$\mathrm{Ca}^{in}(X;B,A)$
(8.2) $^+\mathrm{Ca}^{ex}(S;p^*A,p^*B) \qquad \Rightarrow$ not-$\mathrm{Ca}^{ex}(X;B,A)$

Hence, the plausible new rule of success, for two-level approximative comparative truthlikeness, viz.

TAC-RS: if $\mathrm{Ca}^{in}(R;pA,pB)$ and $\mathrm{Ca}^{ex}(S;p^*A,p^*B)$, and at least once "+", then choose A among A and B,

is again functional for approaching the truth in the sense that the rejected theory B cannot be closer to X than the, for the time being, accepted theory A. Moreover, the rule is again non-frustrating: it cannot prescribe the opposite choice on the basis of new evidence R'/S', $R \subseteq R' \subseteq X_o \subseteq S' \subseteq S$, it can at most change the original preference into indifference.

Again it is easy to formulate evidence-saving theories. For $F \subseteq M_o$, the following definition is standard: $p^{-1}F =_{df} \{x \epsilon M_t / p(x) \epsilon F\}$. It is easy to check that $pp^{-1}F=F=p^*p^{-1}F$ and hence that if $R \subseteq F \subseteq S$, then $p^{-1}F$ (including $p^{-1}R$ and $p^{-1}S$) is as evidence-saving as F (including R and S). However, if we have good reasons to assume that X_t is not equal to, but a proper subset of $p^{-1}X_0$ we are really interested in the theoretical level, and not only in trivial expansions like $p^{-1}F$. But then it is in general not easy to formulate theories that are evidence-saving. Hence, in this situation the whole truthlikeness-machinery becomes relevant.

It is important to note that I am assuming here a realist, even a modal-realist, version of structuralism. From an instrumentalist point of view X_t is just a fiction, only X_0 has some real background, though probably not enough to justify modal terminology even at this level. However this may be, from the instrumentalist point of view, evidence-saving theories are ideal, and the only purpose of proper theories at the theoretical level is predictive power at the observational level.

As to the pretension of a sophisticated structuralist treatment of truthlikeness, one other remark should be made. Besides structures at two levels, an important ingredient of structuralism is the notion of a constraint. This notion, however, was not included in our analysis.

4. *Generality of the analysis*

At first sight it may seem that the treatment of stratified theories is rather restricted in its possible applications. For I did assume a common observational level as well as a common theoretical level. In this final section I shall make some investigations into the generality of the analysis, and shall define the main concepts to discuss this.

I start with some assumptions underlying the treatment of just one level.

M will be called a *structure-set* if M is the set of all structures of a certain type. The first, and fundamental, assumption is:

Ass.1: If a structure-set M represents a set of *conceptual possibilities* of a certain context, then there is a unique subset X of M representing the *empirical possibilities* of the context, relative to M.

From now on everything is related to a certain fixed context.

$<M,X,c>$ is an empirical pseudo-level or, simply, a *pseudo-level* if M is a structure-set representing a set of *conceptual possibilities*, X indicates the (unknown) subset of M representing the *empirical possibilities* relative to M, and c is a (ternary) *structurelikeness* relation (i.e. reflexive, transitive and strongly self-similar) on M.

Pseudo-level $<M',X',c''>$ is an *extension* of $<M,X,c>$ if $M' \supseteq M$, $X' \supseteq X$, $c' \upharpoonright M = c$. Think e.g. of a function in M-structures, which is broadened to a relation in M'-structures.

Pseudo-level $<M,X,c>$ is an empirical level or, simply, a *level* if every extension $<M',X',c'>$ is such that $X'= X$, i.e. extensions do not introduce new empirical possibilities. Note that this implies that every extension of a level is also a level.

The following assumption seems plausible:

Ass. 2: Every pseudo-level can be extended to a (proper) level.

Now it is possible to prove an important theorem, saying that, if M is large enough to give rise to a proper level, its precise scope does no longer matter for internal truthlikeness comparisons, while external comparisons are at least preserved in case of extension.

Th. 3 If level $<M',X,c'>$ is an extension of level $<M,X,c>$, then for every $A,B \subseteq M$

3.1 $Ca^{in}(X;A,B) \Leftrightarrow {'Ca^{in}}(X;A,B)$

3.2 $Ca^{ex}(X;A;B) \Rightarrow {'Ca^{ex}}(X;A,B)$

Of course, 'Ca is defined as Ca (see (i) and (ii), Section 2) with $M=M'$ and $c= c'$. The proof of Th. 3.1 is straightforward. As regards to Th. 3.2 it is, firstly, not difficult to check that equivalence is only prevented by the following subclaim of $Ca^{ex}(X;A,B)$:

$$\forall x \epsilon (A \cap B)\text{-}X\ \forall y \epsilon (A \cap X)\text{-}B \exists z \notin A\ c(x;z,y),$$

but, secondly, that this subclaim, if true for a level, remains trivially true in case of extension, due to the simple fact that the complement of A grows by extension.

From Th. 3 we learn that two theories of the same level, if not yet comparable (in the sense of Ca(X;A,B) or Ca(X;B,A)) can become (and then remain) comparable by (horizontal) extension of the level.

Another question is under what conditions two theories, prima facie formulated at different levels, can be reformulated, without essentially changing them, at the same, possibly a third, level. It will be shown that this is always possible, though this may be in a rather trivial sense. The concepts to be introduced, will however also be useful for still other questions.

We start by defining a plausible quasi-ordering (reflexive and transitive) between levels.

$L_1= <M_1,X_1,c_1>$ is a *superlevel* of $L_2= <M_2,X_2,c_2>$, and L_2 a *sublevel* of L_1 if

1) M_2 can be seen as the projection of M_1 resulting from some (uniform) projection-function p_{12}: $M_1 \rightarrow M_2$, hence, $p_{12}(M_1)=M_2$ (onto),
2) *c-conservation*: $c_1(x;y,z) \Rightarrow c_2(p_{12}(x);p_{12}(y),p_{12}(z))$

Now it is plausible to assume:

Ass.3: If L_1 is a superlevel of L_2 then there is *X-conservation*: $p_{12}(X_1)= X_2$

Next, we have to relativize the definition of a theory: *a theory A of/at level L =* $<M,X,c>$ is the combination of a subset A of M and the claim "$A= X$".

Now, theory A_1 of level L_1 *reduces to* theory A_2 of level L_2 if L_2 is a sublevel of L_1 and $p_{12}(A_1)= A_2$.

If theory A_1 of level L_1 reduces to A_2 of L_2 then X-conservation (Ass.3) guarantees that the "projection" of the claim of theory A_1 of L_1, viz. $X_1= A_1$, onto L_2, viz. $p_{12}(X_1)= p_{12}(A_1)$, is equivalent to the claim of theory A_2 at L_2, viz. $X_2= A_2$. Note also that this includes as special case that theory X_1 of L_1 reduces to theory X_2 of sublevel L_2.

Theory A_1 of level L_1 *reproduces* theory A_2 of level L_2 if A_1 of L_1 reduces to A_2 of L_2 and if moreover $p_{12}^{-1}(A_2)= A_1$.

A direct consequence of these definitions is that if L_2 is a sublevel of L_1 and A_2 a theory at L_2 then $p_{12}^{-1}(A_2)$ reproduces A_2 at L_1.

An indirect consequence is that if the levels of two theories have a common superlevel than these theories can both be reproduced at this common level, and hence they can be subjected to truthlikeness comparisons.

Now the following theorem is easy to prove

Th. 4 For all levels L_2 and L_3 there is a common superlevel, say L_1.

For the proof it is crucial to see that the Cartesian product M_1 of the structure-sets M_2 and M_3 is a new structure-set and that a structurelikeness relation c_1 on M_1 can already be obtained by simply expanding the comparisons based on c_2 and c_3. From Ass.1 it follows that a unique X_1 exists, from Ass.2 that the resulting pseudo-level can be extended to a proper level. For the sake of convenience I shall assume that $<M_1,X_1,c_1>=_{df} L_1$ is already a proper level. Hence, it follows that L_2 and L_3 are sublevels of L_1, q.e.d.

From this proof it is clear that Th. 4 is essentially trivial. However, in practice the question is whether there is a less trivial common level, in which the structures of M_2 and M_3 share one or more components. It is however easy to check that the same device as in the proof of Th. 4 can be applied to the non-overlapping components. Hence, two theories can always be reproduced at a common level.

Let us now turn our attention to the following fact. If theory A_1 of L_1 reproduces A_2 of L_2, theory A_1 is nevertheless not equivalent to theory A_2 in the following sense: someone defending theory A_2 of level L_2 need not to defend that X_1 of superlevel L_1 is $p_{12}^{-1}(X_2)$. One may leave room for the possibility that X_1 is a proper subset of $p_{12}^{-1}(X_2)$, of course, respecting X-conservation, i.e. $p_{12}(X_1)= X_2$, but this is perfectly consistent. In other words, someone defending theory A_2 at level L_2 is only committed to the claim at level L_1 that $p_{12}^{-1}(A_2)= p_{12}^{-1}(X_2)$ and not to the claim that $p_{12}^{-1}(A_2)= X_1$.

By consequence, theory A_1 is only equivalent to theory A_2 if $p_{12}^{-1}(X_2)= X_1$, i.e. if all conceptually possible expansions of members of X_2 belong to X_1. It is clear that this additional condition is essentially a matter of levels. More formally, level L_1 is said to be *restrictable* to sublevel L_2 if $p_{12}^{-1}(X_2)=X_1$; level L is said to be *unrestrictable* if L has no sublevels to which it is restrictable; level L is *exhaustive* if every superlevel of L can be restricted to L. Finally, level L is *optimal* if L is unrestrictable and exhaustive.

Now, theory A_1 of level L_1 is *restrictable* to theory A_2 of sublevel L_2 if L_1 can be restricted to L_2, and if A_1 reproduces A_2. In this situation theories A_1 and A_2 are essentially equivalent.

Let us now try to describe the standard situation of theorizing in terms of the concepts that have been introduced. In the context of interest there is agreement about what, prima facie, to choose as basic or *observational* level $L_o= \langle M_o,X_o,c_o \rangle$. Of course, we only assume that M_o is known and that c_o is defined. The task on this level is to characterize X_o.

Every superlevel of L_o is of course going to be called a *theoretical* level. The first question seems, however, whether L_o is restrictable. Although successful restriction-attempts can make things conceptually much easier, there is no fundamental reason to answer the first question first.

The second and main question is of course whether there is an exhaustive superlevel. It is easy to see that if there are two exhaustive superlevels, the one not just an extension of the other, then one of them should be a sublevel of the other, to which the latter can be restricted, for otherwise it is easy to construct a third superlevel which cannot be restricted to any of the two, which would make the latter non-exhaustive, contrary to the initial assumption. Hence, if we believe to have found an exhaustive level, say L_t, including a characterization of X_t, we have almost everything we could wish, in particular, we have not only a characterization of X_o (viz. $p_{to}(X_t)$), but a characterization of the X of all sublevels of L_t. The only remaining question is whether L_t cannot or can be restricted, i.e. whether it is or is not optimal. If it can be restricted, it may or may not imply that L_o could already have been restricted.

It is tempting to raise now the question whether there is, starting from any observational level, always an exhaustive or even an optimal superlevel. In classical terms, it is the question whether there exists for every observational problem area an ideal language. Unfortunately, it is difficult to answer this question. However this may be, I believe at least that optimal levels can exist.

I conclude with a byproduct of the foregoing analysis: a characterization of a paradigm or researchprogramme. Relative to some observational level L_o there is a (proper) superlevel L_C, called the *core level*, and a *core theory* C at this level (which is not a reproduction of a "lower theory"). All specific theories T in the researchprogramme belong to L_C or a higher level, say L_T, and are such that $p_{TC}(T) \subseteq C$, i.e. they obey the hard core.